Spin-wave theory for the biaxial (m=2) Lifshitz point problem in three dimensions

1980 J. Phys. A: Math. Gen. 13 L201

(http://iopscience.iop.org/0305-4470/13/6/008)

J. Phys. A: Math. Gen., 13 (1980) L201–L205. Printed in Great Britain

LETTER TO THE EDITOR

Spin-wave theory for the biaxial ($m = 2$) Lifshitz point problem in three dimensions

G Grinstein
IBM T J Watson Research Center, Yorktown Heights, NY 10598, USA

Received 19 February 1980

Abstract. It is known that no long-range order can exist on the boundary between the helical phase (wherein the magnetisation varies spatially in one or more of m distinct directions) and the ferromagnetic phase in the biaxial ($m = 2$) Lifshitz point problem in three dimensions when n, the number of components of the order parameter, is greater than unity. The *Gaussian* (quadratic) spin-wave approximation to the $n = 2$ problem predicts that on this phase boundary correlations decay as power laws at large distance. It is shown here that the presence of a *marginal* quartic spin-wave operator produces logarithmic corrections to the power laws.

Mermin and Wagner (1966) have proved that there can be no long-range order in the two-dimensional (2D) XY model. The 'Gaussian spin-wave' (SW) approximation to the model nonetheless predicts power law decay of spin correlation functions at all temperatures (Wegner 1967, Berezinskii 1971, Zittartz 1976a, b), rather than the exponential decay which typically characterises a disordered phase. Since the SW theory is presumably an excellent approximation at low temperatures, these power laws strongly suggested that the 2D XY model has an unorthodox phase at low T. Kosterlitz and Thouless (1973) and Kosterlitz (1974) subsequently showed that the model indeed undergoes a continuous transition from an orthodox paramagnetic state at high T to a state with algebraically decaying correlations but no long-range order at low T.

It has been appreciated for some time (Grest and Sak 1978) that the model describing the '$m = 2$, $n = 2$ Lifshitz point (22LP)' (Hornreich *et al* 1975) is a 3D analogue of the 2D XY model, in that on the one hand it cannot exhibit long-range order, while on the other hand calculations in $3+\epsilon$ dimensions (Grest and Sak 1978) suggest that the 3D problem has a finite critical temperature T_c.

In this Letter we study the SW theory for this 3D problem. We show that the long-distance behaviour of correlation functions is, in contrast to the 2D XY case, not completely described by the Gaussian SW approximation, even at very low T. There is, in addition to the *marginal quadratic* SW operators familiar from the 2D XY problem, a *marginal quartic* operator in the 22LP SW theory. Standard renormalisation group (RG) analysis of this marginal operator gives, as usual (Wegner and Riedel 1973), logarithmic corrections to the power laws predicted by the Gaussian SW theory. That is, the spin–spin correlation function $G(r)$ behaves like $r^{-\eta(T)}(\log r)^{-\tilde{\eta}(T)}$ for large r; η and $\tilde{\eta}$ are *non-universal* functions of T in that they depend on the details of the short-distance cut-off used to ensure the finiteness of the theory.

0305-4470/80/060201+05\$01.50

The general 'm, n, d' LP problem is described (Hornreich *et al* 1975) by the free energy functional

$$\tilde{H}=\frac{J}{2}\int \mathrm{d}^d x[(\nabla_\perp \boldsymbol{S})^2+A(\nabla_\| \boldsymbol{S})^2+(\nabla_\|^2 \boldsymbol{S})^2]. \tag{1}$$

Here J is the exchange strength, the parameter A will assume poth positive and negative values, and the spin $\boldsymbol{S}$ has n components and unit magnitude: $\boldsymbol{S}^2=1$. Of the d spatial dimensions, m are arbitrarily denoted 'parallel', and the remaining $d-m$ 'perpendicular'. That is,

$$(\nabla_\|\boldsymbol{S})^2\equiv\sum_{\alpha=1}^{n}\sum_{i=1}^{m}(\nabla_i S_\alpha)^2, \qquad (\nabla_\perp\boldsymbol{S})^2\equiv\sum_{\alpha=1}^{n}\sum_{i=m+1}^{d}(\nabla_i S_\alpha)^2,$$

$$(\nabla_\|^2\boldsymbol{S})^2\equiv\sum_{\alpha=1}^{n}\left(\sum_{i=1}^{m}\nabla_i^2 S_\alpha\right)^2.$$

Hamiltonian $\tilde{H}$ gives rise to the phase diagram shown schematically in figure 1. The LP is the critical point connecting the paramagnetic ($\langle\boldsymbol{S}\rangle=0$), ferromagnetic ($\langle\boldsymbol{S}(\boldsymbol{q}=0)\rangle\neq 0$) and helical ($\langle\boldsymbol{S}(q_\perp=0,\ q_\|\sim|A|^{1/2})\rangle\neq 0$) phases.

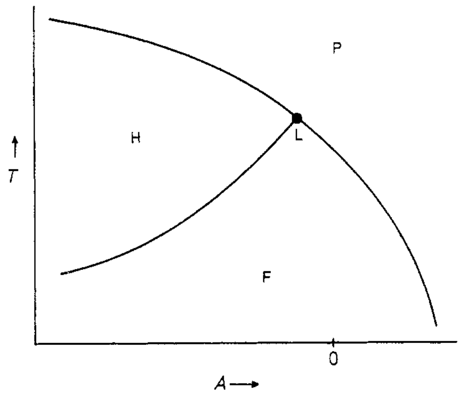

Figure 1. Schematic Lifshitz point phase diagram as a function of temperature T and the parameter A. The paramagnetic (P), ferromagnetic (F) and helical (H) phases meet at the Lifshitz point (L). The present Letter is concerned with the ferromagnetic–helical phase boundary.

In the ferromagnetic domain of figure 1 the transverse spin–spin correlation function for $n\geqslant 2$ behaves roughly like $[q_\perp^2+Dq_\|^2+(q_\|^2)^2]^{-1}$ for small q, where D is a positive function of A and T. As the helical region is approached, D decreases, vanishing on the phase boundary between the helical and ferromagnetic regions. On this boundary, then, the assumption of ferromagnetic order implies a transverse spin correlation function which in position space behaves like $\int \mathrm{d}^{d-2}q_\perp\,\mathrm{d}^2q_\|/[q_\perp^2+(q_\|^2)^2]$ near $\boldsymbol{q}=0$ for $m=2$ (the biaxial case), and therefore diverges for $d\leqslant 3$. Thus for $n\geqslant 2$ and $m=2$ there cannot be long-range order on the boundary. (The possibility that a *first-order* ferromagnetic–helical transition occurs for some positive D, thereby invalidating this argument, cannot be ruled out. In this Letter we consider the consequences of a *continuous* disappearance of ferromagnetic order.)

Grest and Sak's (1978) $(3+\epsilon)$-dimensional calculations for the $m=2$ LP suggest, in analogy with $2+\epsilon$ expansions for the n-vector model (Migdal 1975, Polyakov 1975, Brézin and Zinn-Justin 1976), that the 22LP problem in 3D has a finite T_c. The low-T phase of this model can be investigated by SW theory, readily constructed by the substitution (Brézin and Zinn-Justin 1976) $S \equiv S_1 + \mathrm{i}S_2 = \mathrm{e}^{\mathrm{i}w\theta}$ in (1); this fulfils the $S_1^2+S_2^2=1$ constraint. The partition function $Z = \mathrm{Tr}\, \mathrm{e}^{-\tilde{H}/k_\mathrm{B}T}$ becomes $\mathrm{Tr}\, \mathrm{e}^{-H}$, where

$$H = \tfrac{1}{2}\int \mathrm{d}^3x\{(\nabla_\perp\theta)^2 + A(\nabla_\parallel\theta)^2 + (\nabla_\parallel^2\theta)^2 + u[(\nabla_\parallel\theta)^2]^2\}, \tag{2}$$

$u = w^2 \equiv k_\mathrm{B}T/J$, and $\theta(\boldsymbol{x})$ runs from $-\infty$ to ∞. This constitutes the SW approximation.

The Gaussian (quadratic) SW approximation is obtained by setting u to zero in the Hamiltonian (2); the LP then occurs at $A=0$. The problem becomes formally identical to SW descriptions of a smectic A liquid crystal. In that context the correlation functions have already been evaluated (Caillé 1972): $\langle S\rangle = 0$ as anticipated, while $G(\boldsymbol{x}) \equiv \langle S(\boldsymbol{x})S(\boldsymbol{0})\rangle$ is a complicated anisotropic function which behaves at large $|\boldsymbol{x}|$ like $x_\parallel^{-k_\mathrm{B}T/4\pi J}$ and $x_\perp^{-k_\mathrm{B}T/8\pi J}$ in the parallel and perpendicular directions respectively.

The $[(\nabla_\parallel\theta)^2]^2$ term can be treated by standard renormalisation group methods. Simple power counting shows (Wegner 1976) that the operator is marginal at all temperatures. To see this explicitly note that if the terms $\int(\nabla_\parallel^2\theta)^2$ and $A\int(\nabla_\parallel\theta)^2$ in (2) are to be properly dimensionless, θ and A must have dimensions $\mu^{-1/2}$ and μ^2 respectively, where μ is a mass. In that case, however, $\int(\nabla_\perp\theta)^2$ has dimension μ^{-2}, and the quadratic part of the Hamiltonian should really be written in the form $\int \mathrm{d}^3y[\mu^2(\nabla_\perp\tilde{\theta})^2 + A(\nabla_\parallel\tilde{\theta})^2 + (\nabla_\parallel^2\tilde{\theta})^2]$. The extraneous mass μ can be eliminated by the transformation $\tilde{\theta}(y_\parallel, y_\perp) = \mu^{-1/2}\theta(\boldsymbol{x}_\parallel, x_\perp)$, $\boldsymbol{y}_\parallel = \boldsymbol{x}_\parallel$, $y_\perp = \mu x_\perp$, whereupon $\mathrm{d}^3y = \mathrm{d}^2y_\parallel\, \mathrm{d}y_\perp = \mu\, \mathrm{d}^3x$ and the Hamiltonian acquires the form (2), with θ now dimensionless and $x_\perp$ having dimension μ^{-2}. It follows at once that the coupling constant u of the $\int[(\nabla_\parallel\theta)^2]^2$ term is dimensionless; this is the classic signature of a marginal operator (Brézin *et al* 1976, Wegner 1976).

It is convenient to employ the techniques of Brézin *et al* (1976) to generate the following RG equation for the spin–spin correlation function $G(\boldsymbol{x}, u, w, \Lambda) = \langle \mathrm{e}^{\mathrm{i}w\theta(\boldsymbol{x})}\, \mathrm{e}^{-\mathrm{i}w\theta(\boldsymbol{0})}\rangle$; here Λ is the high-momentum cut-off for the theory, and for simplicity we set $\boldsymbol{x} = (\boldsymbol{x}_\parallel, x_\perp = 0)$:

$$\left(\frac{\partial}{\partial l} + \beta_w(u,w)\frac{\partial}{\partial w} + \beta_u(u)\frac{\partial}{\partial u} + \gamma(u,w)\right)G(x_\parallel) = 0, \tag{3a}$$

$$\beta_w(u,w) = c_1wu^2 + \mathrm{O}(wu^3), \tag{3b}$$

$$\beta_u(u) = 9u^2/8\pi + \mathrm{O}(u^3), \tag{3c}$$

$$\gamma(u,w) = w^2/4\pi + c_2uw^4 + \mathrm{O}(u^2). \tag{3d}$$

Here $l \equiv \log\Lambda$, $c_1 = [(\log\frac{4}{3}) - \frac{5}{48}]/2\pi^2$, $c_2 = (\log\frac{32}{27})/(4\pi)^3$ and the higher-order terms on the right-hand sides of (3b)–(3d) have coefficients which are of course non-universal: they depend on the details of the high-momentum cut-off prescription. In the derivation of these equations A has been chosen as a function of u to locate the ferromagnetic–helical phase boundary; the equations are therefore independent of A. Note that G, being dimensionless, is really a function only of the three variables $(x_\parallel\Lambda)$, u and w.

The solution of (3a) is well known (Amit 1978); for large l we have

$$G(x_\parallel, u, w, \Lambda) \sim \exp\left(-\int_0^l \gamma(\tilde{u}, \tilde{w})\, d\tau\right), \tag{4}$$

where $\tilde{u}(u, \tau)$ and $\tilde{w}(u, w, \tau)$ are determined by the equations $\partial\tilde{u}/\partial\tau = -\beta_u(\tilde{u})$, $\partial\tilde{w}/\partial\tau = -\beta_w(\tilde{u}, \tilde{w})$ and the boundary conditions $\tilde{u}(u, 0) = u$, $\tilde{w}(u, w, 0) = w$. The marginality of the $[(\nabla_\parallel\theta)^2]^2$ operator is manifest (Wegner and Riedel 1973) in the absence of a term linear in $\tilde{u}$ (see (3c)) on the right-hand side of the $\tilde{u}$ equation, which for small u has the solution

$$\tilde{u} = (u^{-1} + 9\tau/8\pi)^{-1}. \tag{5}$$

(It is simple to verify that all powers of $\nabla_\parallel\theta$ and $\nabla_\perp\theta$ higher than those in Hamiltonian (2) are *irrelevant* operators: their inclusion in (2) does not alter the large-distance behaviour of correlation functions.)

The solution of the $\tilde{w}$ equation follows from (3b) and (5):

$$\tilde{w} = w \exp[(-c_1 u^2 \tau)(1 + 9u\tau/8\pi)^{-1}]. \tag{6}$$

Substitution of (5) and (6) into (4), and integration then yields for large $x_\parallel$

$$G(x_\parallel) \sim x_\parallel^{-\eta(T)} (\log x_\parallel)^{-\tilde{\eta}(T)}, \tag{7a}$$

where

$$\eta(T) = (w^2/4\pi)(1 - 16\pi c_1 u/9 + O(u^2)), \tag{7b}$$

$$\tilde{\eta}(T) = (128\pi^2/9)(c_1\eta(T)/9 + \pi c_2 \eta^2(T)), \tag{7c}$$

and $u = w^2 = k_B T/J$. For $x_\parallel = 0$ and finite $x_\perp$ one similarly obtains (7a) with $x_\parallel$ replaced by $x_\perp^{1/2}$. Note that while the complete function η (and hence $\tilde{\eta}$) is non-universal, the two terms explicitly displayed in (7b) are universal (that is, cut-off-independent). As $T \to 0$, η approaches the limit $k_B T/4\pi J$.

The SW theory predicts logarithm-corrected power laws at all temperatures. It is too crude to provide any information about the Lifshitz point; that is, about the transition into the paramagnetic phase where correlations decay exponentially. It is, however, reliable at low temperatures along the line separating the helical and ferromagnetic phases. Presumably the logarithm-corrected power laws persist right up to the Lifshitz point along that line.

Although the SW description of a smectic A liquid crystal (Caillé 1972) is similar to the SW theory studied here, the $(\nabla_\parallel\theta)^2$ and $[(\nabla_\parallel\theta)^2]^2$ terms are prevented (Landau and Lifshitz 1958) by symmetry from entering the liquid crystal Hamiltonian. No logarithmic corrections mar the algebraic decay in a smectic A (Als-Nielsen *et al* 1977) at large distance.

The SW approximation to the 22LP problem in a magnetic field is described by a 3D anisotropic sine–Gordon equation, which, like its 2D isotropic counterpart (Coleman 1975, José *et al* 1977, Wiegmann 1978, Ohta 1978, Amit *et al* 1979) exhibits an infinite-order (Kosterlitz 1974, Wegner 1976) phase transition. Details will be given elsewhere.

I thank A Aharony, J Als-Nielsen, B Friman, P Horn, Y Imry, C Jayaprakash, L Kadanoff, S Kirkpatrick, R Pelcovits and E Pytte for helpful conversations, and C Jayakaprash and S Kirkpatrick for critical reading of the manuscript.

Note added in proof. T A Kaplan (1980 *Phys. Rev. Lett.* **44** 760) has independently studied the Gaussian spin wave theory for the $m = 2$ $d = 3$ LP problem.

References

Als-Nielsen J, Birgeneau R J, Kaplan M, Litster J D and Safinya C R 1977 *Phys. Rev. Lett.* **39** 1668
Amit D J 1978 *Field Theory, The Renormalization Group and Critical Phenomena* (New York: McGraw-Hill)
Amit D J, Goldschmidt Y Y and Grinstein G 1979 *Preprint*
Berezinskii V L 1971 *Zh. Eksp. Teor. Fiz.* **59** 907 (Engl. transl. 1972 *Sov. Phys.–JETP* **32** 493)
Brézin E, Le Guillou J C and Zinn-Justin J 1976 in *Phase Transitions and Critical Phenomena* vol 6 (London: Academic)
Brézin E and Zinn-Justin J 1976 *Phys. Rev. Lett.* **36** 691; *Phys. Rev.* B **14** 3110
Caillé A 1972 *C. R. Acad. Sci. Paris* B **274** 891
Coleman S 1975 *Phys. Rev.* D **11** 2088
Grest G S and Sak J 1978 *Phys. Rev.* B **17** 3607
Hornreich R M, Luban M and Shtrikman S 1975 *Phys. Rev. Lett.* **35** 1678
José J V, Kadanoff L P, Kirkpatrick S and Nelson D R 1977 *Phys. Rev.* B **16** 1217
Kosterlitz J M 1974 *J. Phys. C: Solid St. Phys.* **7** 1046
Kosterlitz J M and Thouless D J 1973 *J. Phys. C: Solid St. Phys.* **6** 1181
Landau L D and Lifshitz E M 1958 *Statistical Physics* (London: Pergamon)
Mermin N D and Wagner H 1966 *Phys. Rev. Lett.* **17** 1133
Migdal A A 1975 *Zh. Eksp. Teor. Fiz.* **69** 1457 (Engl. transl. *Sov. Phys.–JETP* **42** 743)
Ohta T 1978 *Prog. Theor. Phys.* **60** 968
Polyakov A M 1975 *Phys. Lett.* B **59** 79
Wegner F J 1967 *Z. Phys.* **206** 465
—— 1976 in *Phase Transitions and Critical Phenomena* vol 6 (London: Academic)
Wegner F J and Riedel E K 1973 *Phys. Rev.* B **7** 248
Wiegmann P B 1978 *J. Phys. C: Solid St. Phys.* **11** 1583
Zittartz J 1976a *Z. Phys.* B **23** 55
—— 1976b *Z. Phys.* B **23** 63